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§ 14.? : Multivariable Chain Rule

Goal: Extend the chain rule from Calculus I to multivariable functions.

> In calc I
>
> $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$
>
> $(f \circ g)(x) = f(g(x))$

Composition of Multivariable Functions

Given a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$

So $f(x_1, x_2, x_{\dots}, x_n)$

To generalize composition of Calculus I, we will allow each coordinate x_i to be a function of other variables... ie. $x_i = g_i(t_1, t_2, \dots, t_k)$

Ex. Let $f(x,y,z) = xy + yz - z^2$ and

$$x(s,t) = s - t,\ y(s,t) = s^2 + t,\ z(s,t) = \cos(t)$$

The composition $f(x(s,t), y(s,t), z(s,t))$ has formula...

$$f(x(s,t), y(s,t), z(s,t)) = f(s-t, s^2+t, \cos(t))$$
$$= (s-t)(s^2+t) + (s^2+t)(\cos(t)) - \cos^2(t)$$

(we can simplify if you want)

Def^n: A function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $p \in D$ when f is "well-approximated" by its tangent (hyper)plane at p.

NB: This notion is (basically) the same notion from Calculus I.

Observation: If $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^k \to \mathbb{R}$ for $1 \le i \le n$, the composition of

$f(g_1(s_1, s_2, \dots, s_k), g_2(s_1, s_2, \dots, s_k), g_3(s_1, s_2, \dots, s_k))$

is a function of k-variables.

$$\begin{tikzcd}
\mathbb{R}^k \arrow[r, "g_1"] \arrow[dr, "g_2"] \arrow[ddr, "g_n"'] & \mathbb{R} \arrow[r, "1^{st}"] & \mathbb{R}^n \arrow[r, "f"] & \mathbb{R} \\
& \mathbb{R} \arrow[ur, "2^{nd}"] & & \\
& \mathbb{R} \arrow[uur, "n^{th}"'] & &
\end{tikzcd}$$

So we have

$$\mathbb{R}^k \xrightarrow{g_i} \mathbb{R}^n \xrightarrow{f} \mathbb{R}$$

- Suppose $f(x,y)$ and $x(t), y(t)$ are differentiable near $p=(a,b)$, $f(x,y) = f(a,b) + (f_x(a,b)+\mathcal{E}_x)(x-a) + (f_y(a,b)+\mathcal{E}_y)(y-b)$
  where $(\mathcal{E}_x, \mathcal{E}_y) \to (0,0)$ as $(x,y) \to (a,b)$
- Let t_0 be a time so that $(x(t_0), y(t_0)) = (a,b)$
  Our tangent plane evaluated along $(x(t), y(t))$ becomes:

$$f(x(t),y(t)) = f(x(t_0),y(t_0)) + \left(f_x(x(t_0),y(t_0)) + \mathcal{E}_x\right)(x(t)-x(t_0)) + \left(f_y(x(t_0),y(t_0)) + \mathcal{E}_y\right)(y(t)-y(t_0))$$

- $\therefore$ $$f(x(t),y(t)) - f(x(t_0),y(t_0)) = f_x(x(t_0),y(t_0))(x(t)-x(t_0)) + f_y(x(t_0),y(t_0))(y(t)-y(t_0)) + \mathcal{E}_x(x(t)-x(t_0)) + \mathcal{E}_y(y(t)-y(t_0))$$
- Now divide both sides by $t-t_0$ when $t-t_0 \neq 0$
- $$\frac{f(x(t),y(t)) - f(x(t_0),y(t_0))}{t-t_0} = f_x(x(t_0),y(t_0))\left(\frac{x(t)-x(t_0)}{t-t_0}\right) + f_y(x(t_0),y(t_0))\left(\frac{y(t)-y(t_0)}{t-t_0}\right) + \mathcal{E}_x\left(\frac{x(t)-x(t_0)}{t-t_0}\right) + \mathcal{E}_y\left(\frac{y(t)-y(t_0)}{t-t_0}\right)$$
- limiting $t \to t_0$, we obtain:

$$\frac{\partial}{\partial t}\left[f(x(t),y(t))\right]\Big|_{t=t_0} = \lim_{t\to t_0} \frac{f(x(t),y(t)) - f(x(t_0),y(t_0))}{t-t_0}$$

- $$= f_x(x(t_0),y(t_0)) \lim_{t\to t_0} \frac{x(t)-x(t_0)}{t-t_0} + f_y(x(t_0),y(t_0)) \lim_{t\to t_0} \frac{y(t)-y(t_0)}{t-t_0} + \lim_{t\to t_0} \mathcal{E}_x \cdot \lim_{t\to t_0} \frac{x(t)-x(t_0)}{t-t_0} + \lim_{t\to t_0} \mathcal{E}_y \cdot \lim_{t\to t_0} \frac{y(t)-y(t_0)}{t-t_0}$$

- $$= f_x(x(t_0),y(t_0))\, x'(t_0) + f_y(x(t_0),y(t_0))\, y'(t_0) + \lim_{t\to t_0} \mathcal{E}_x \cdot x'(t_0) + \lim_{t\to t_0} \mathcal{E}_y \cdot y'(t_0)$$

... Continued on next page.

As $t \to t_0$, $(x(t), y(t)) \to (a,b)$, so $(\mathcal{E}_x(t), \mathcal{E}_y(t)) \to (0,0)$

hence $\boxed{\frac{\partial}{\partial t}\left[f(x(t),y(t))\right]\Big|_{t=t_0} = f_x(x(t_0),y(t_0))\,x'(t_0) + f_y(x(t_0),y(t_0))\,y'(t_0)}$

The derivation we just performed can be generalized to prove:

<u>Prop (Multivariate Chain Rule)</u>: Suppose $f(x_1, x_2, \dots, x_n)$ and $x_i = x_i(t_1, t_2, \dots, t_k)$ are differentiable. Then

$$\frac{\partial f}{\partial t_j} = \frac{\partial f}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_j} + \frac{\partial f}{\partial x_2}\cdot\frac{\partial x_2}{\partial t_j} + \dots + \frac{\partial f}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_j}$$

<u>Ex.</u> Compute $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}$ for

$$f(x,y,z) = x^4 y + y^2 z^3, \quad x = rse^t, \; y = rs^2e^{-t}, \; z = r^2 s\sin(t)$$

<u>Sol #1</u>: (Not using chain Rule)

$$\begin{aligned} f(x,y,z) &= f(rse^t, rs^2e^{-t}, r^2 s\sin(t)) \\ &= (rse^t)^4(rs^2e^{-t}) + (rs^2e^{-t})^2(r^2 s\sin(t))^3 \\ &= r^5 s^6 e^{3t} + r^8 s^7 e^{-2t}\sin^3(t) \end{aligned}$$

$$\frac{\partial f}{\partial r} = 5r^4 s^6 e^{3t} + 8r^7 s^7 e^{-2t}\sin^3(t)$$

$$\frac{\partial f}{\partial s} = 6r^5 s^5 e^{3t} + 5r^8 s^6 e^{-2t}\sin^3(t)$$

$$\frac{\partial f}{\partial t} = 3r^5 s^6 e^{3t} + r^8 s^7\left(-2e^{-2t}\sin^3(t) + e^{-2t}\cdot 3\sin^2(t)\cos(t)\right)$$

<u>Sol #2</u>: (Using the Chain Rule)

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$\frac{\partial f}{\partial x} = 4x^3 y = 4(rse^t)^3(rs^2e^{-t}) = 4r^4 s^5 e^{2t}$$

$$\frac{\partial f}{\partial y} = x^4 + 2yz^3 = (rse^t)^4 + 2(rs^2e^{-t})(r^2 s\sin(t))^3 = r^4 s^4 e^{4t} + 2r^7 s^5 e^{-t}\sin^3(t)$$

$$\frac{\partial f}{\partial z} = 3y^2 z^2 = 3(rs^2e^{-t})^2(r^2 s\sin(t))^2 = 3r^6 s^6 e^{-2t}\sin^2(t)$$

$\frac{\partial x}{\partial r} = se^{t}$, $\frac{\partial y}{\partial r} = s^2 e^{-t}$, $\frac{\partial z}{\partial r} = 2rs\sin(t)$

$$\frac{\partial F}{\partial r} = 4r^4 s^5 e^{2t} \cdot se^{t} + \left(r^4 s^4 e^{4t} + 2r^7 s^5 e^{-t}\sin^3(t)\right) \cdot s^2 e^{-t} + 3r^6 s^6 e^{-2t}\sin^2(t) \cdot 2rs\sin(t)$$

$$= 4r^4 s^6 e^{3t} + r^4 s^6 e^{3t} + 2r^7 s^7 e^{-2t}\sin^3(t) + 6r^7 s^7 e^{-2t}\sin^3(t)$$
$$= 5r^4 s^6 e^{3t} + 8r^7 s^7 e^{-2t}\sin^3(t)$$

To compute $\frac{\partial F}{\partial s}$, $\frac{\partial x}{\partial s} = re^{t}$, $\frac{\partial y}{\partial s} = 2rse^{-t}$, $\frac{\partial z}{\partial s} = r^2\sin(t)$

$$\therefore \frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$$= \left(4r^4 s^5 e^{2t}\right)\left(re^{t}\right) + \left(r^4 s^4 e^{4t} + 2r^7 s^5 e^{-t}\sin^3(t)\right)\left(2rse^{-t}\right) + \left(3r^6 s^6 e^{-2t}\sin^2(t)\right)\left(r^2\sin(t)\right)$$

To compute $\frac{\partial F}{\partial t}$, $\frac{\partial x}{\partial t} = rse^{t}$, $\frac{\partial y}{\partial t} = -rs^2 e^{-t}$, $\frac{\partial z}{\partial t} = r^2 s\cos(t)$

$$\therefore \frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial t}$$

$$= \left(4r^4 s^5 e^{2t}\right)\left(rse^{t}\right) + \left(r^4 s^4 e^{4t} + 2r^7 s^5 e^{-t}\sin^3(t)\right)\left(-rs^2 e^{-t}\right) + \left(3r^6 s^6 e^{-2t}\sin^2(t)\right)\left(r^2 s\cos(t)\right)$$

<u>Exercise</u>: Repeat (w/both solutions) for

$F(x,y) = e^{x}\sin(y)$, $x = st^2$, $y = s^2 t$

Find $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$

Q: Given an implicit (hyper)surface, how do we compute the slope of the tangent at a given point?

A: Use Implicit Function Theorem (IFT)

Prop (Implicit Function Theorem): Suppose $F(x_1, x_2, \dots, x_n)$ is differentiable on a disk containing $\vec{p}$ and $F(\vec{p}) = 0$ and $\frac{\partial F}{\partial x_i}$ are continuous and $\left.\frac{\partial F}{\partial x_n}\right|_{\vec{p}} \neq 0$

Then, near $\vec{p}$, $x_n = f(x_1, x_2, \dots, x_{n-1})$ and for all i:

$$\frac{\partial x_n}{\partial x_i} = -\frac{\partial F}{\partial x_i} \Big/ \frac{\partial F}{\partial x_n}$$

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Pf (of Derivative formula): $x_n = f(x_1, x_2, \dots, x_{n-1})$

Apply the chain rule to compute $\frac{\partial F}{\partial x_i}$

$$0 = \frac{\partial F}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_i} + \frac{\partial F}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_i} + \cdots + \frac{\partial F}{\partial x_n} \cdot \frac{\partial x_n}{\partial x_i}$$

Unless $i = k$ or $i = n$, $\frac{\partial x_k}{\partial x_i} = 0$, so we see

$$0 = \frac{\partial F}{\partial x_i} \cdot \underbrace{\frac{\partial x_i}{\partial x_i}}_{1} + \frac{\partial F}{\partial x_n} \cdot \frac{\partial f}{\partial x_i}$$

$$-\frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_n} \cdot \frac{\partial f}{\partial x_i}$$

$$\boxed{\frac{\partial f}{\partial x_i} = -\frac{\partial F}{\partial x_i} \Big/ \frac{\partial F}{\partial x_n}}$$